Nonthermal nature of extremal Kerr black holes

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Abstract

Liberati, Rothman and Sonego have recently showed that objects collapsing into extremal Reissner-Nordström black holes do not behave as thermal objects at any time in their history. In particular, a temperature, and hence thermodynamic entropy, are undefined for them. I demonstrate that the analysis goes through essentially unchanged for bodies collapsing into extremal Kerr black holes.

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1 Introduction

Recent results in quantum field theory in curved spacetime have demonstrated fairly conclusively that extremal Reissner-Nordström (RN) black holes are of a fundamentally different class than their nonextremal counterparts. Liberati, Rothman and Sonego (LRS, [1]) examined spherical bodies collapsing into extremal Reissner-Nordström black holes and showed that the object's particle-emission spectrum is nonthermal. Consequently a temperature is undefined for them. Furthermore, at no time during the history of the object do the properties of the stress-energy tensor of the radiation correspond to those of a nonextremal RN hole, and thus extremal holes never behave as thermal objects. Simultaneously with the LRS work, Anderson, Hiscock and Taylor [2] demonstrated that static extremal RN solutions for "physically realistic" values of the curvature coupling constant simply do not exist to perturbative order in semi-classical gravity. Taken together, these results and others (see LRS for references) constitute strong evidence that the $Q^2 = M^2$ RN solutions, which are the absolute-zero state in black hole dynamics, are for reasons as yet unclear disallowed by nature.

Of greater astrophysical interest than charged black holes are rotating black holes. For that reason it is important to extend the results to the extremal Kerr metric. It is well

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known (see, e.g. Hawking [3] or DeWitt [4]) that the spectrum of the nonextremal Kerr solution can be obtained from the Schwarzschild-Droste (SD)[1] spectrum merely by taking into account the frequency shift induced by the rotation of the hole (below). One suspects that the extremal Kerr solution can be obtained from the extremal RN solution in the same way but because previous work has shown that extremality is dangerous territory, one might want to see this as explicitly as possible. In this letter I demonstrate that, to be sure, the behavior of the extremal Kerr solution is essentially identical to that of the RN case. Thus it appears that the nonthermal nature of extremal black holes is a generic property. In retrospect, this is to have been expected because the nature of the horizon changes completely at extremality [5].

LRS used the well known "moving mirror" technique to model the collapse. Rather than deal with the full four-dimensional problem one considers a one-dimensional mirror moving in two-dimensional Minkowski space. Massless scalar fields are propagated inwards from $\mathcal{I}^-$ and bounced off the mirror to $\mathcal{I}^+$. Due to the mirror's motion, one expects that the In and Out vacuum states will differ, leading to particle production. The spectrum is a function of the mirror's trajectory, and so a key step in the calculation is to determine the worldline of the star's center just before formation of the horizon. This late-time trajectory is taken in turn to correspond to the asymptotic trajectory of the mirror. Unfortunately, Kruskal corrdinates, which are typically employed in such calculations and which contain the surface gravity κ, break down in the extremal limit where κ vanishes (below). LRS were able to provide a simple coordinate extension for this case, which allowed them to calculate the late-time history of the center of the collapsing object. Remarkably, this turned out to correspond precisely to that of a uniformly accelerated mirror in Minkowski space. The properties of such a system are well known [6]. In particular, the emission spectrum is nonplanckian, demonstrating that temperature is undefined for these objects. I now show that the same applies to the Kerr solution.

2 Coordinate extension for the extremal Kerr solution

The Kerr metric in Kerr coordinates is

$$\begin{aligned} ds^2 &= -[1-2Mr\rho^{-2}]dv^2 + 2drdv + \rho^2 d\theta^2 + \rho^{-2}[(r^2+a^2)^2 \\ &\quad -\Delta a^2 sin^2\theta]sin^2\theta d\phi_*^2 - 2asin^2\theta d\phi_* dr - 4Mra\rho^{-2}sin^2\theta d\phi_* dv, \end{aligned} \tag{2.1}$$

where as usual

$$dv = dt + \frac{(r^2+a^2)dr}{\Delta}\ ;\ d\phi_* = d\phi + \frac{adr}{\Delta} \tag{2.2}$$

and

$$\rho^2 = r^2 + a^2 cos\theta\ ;\ \Delta = r^2 - 2Mr + a^2\ . \tag{2.3}$$

[1]Johannes Droste, a pupil of Lorentz, independently announced the "Schwarzschild" exterior solution within four months of Schwarzschild. See, "The field of a single centre in Einstein's theory of gravitation, and the motion of a particle in that field," Koninklijke Nederlandsche Akademie van Wetenschappen, Proceedings **19**, 197 (1917).

Note that dv represents an advanced null coordinate corresponding to the $dv = dt + dr_*$ employed in SD or RN calculations, and that a is the angular momentum parameter.

To find the late-time history of the collapsing body's center, it is useful to employ double-null coordinates and so we define

$$du = dt - \frac{(r^2+a^2)dr}{\Delta} \tag{2.4}$$

Before rewriting the Kerr metric in terms of u, v, it is important to verify that these coordinates are continuous in the extremal limit $a = M$. This is easily done. Integrating the expression for dv in (2.2) yields for the nonextremal case:

$$\begin{aligned} v(a,M) &\equiv t + r_*(a,M) \\ &= t + \tfrac{1}{2\sqrt{M^2-a^2}}\{(r_+ - r_-)r + (r_+^2 + a^2)\ln(r - r_+) \\ &\quad -(r_-^2 + a^2)\ln(r - r_-)\} + \text{const} . \end{aligned} \tag{2.5}$$

Here, $r_\pm \equiv M \pm \sqrt{M^2 - a^2}$.

The extremal value for v is found by setting $a = M$ in the expression for dv *before* integrating. Then $\Delta = (r - M)^2$ and

$$dv(M,M) = dt + \frac{(r^2+M^2)dr}{(r-M)^2}. \tag{2.6}$$

Integrating gives

$$\begin{aligned} v(M,M) &\equiv t + r_*(M,M) \\ &= t + r + 2M\ln(r-M) - \frac{2M^2}{(r-M)} + \text{const.} \end{aligned} \tag{2.7}$$

Note that (2.5) appears to give $0/0$ when $a = M$, whereas (2.7) is well behaved except at $r = M$, the extremal horizon. Nevertheless, if one sets $M = a + \epsilon,\ \epsilon << a$ and works to first order in ϵ, it is straightforward to show that (2.7) is the limit of (2.5) as $\epsilon \to 0$. Therefore, the coordinate v is continous at extremality. The same holds for u and ϕ_*.

Since u and v are good at extremality, we now specialize to the case $a = M$. The metric (2.1) becomes

$$\begin{aligned} ds^2 &= -[1 - \frac{2Mr}{\rho^2}]dv^2 + 2drdv + \rho^2 d\theta^2 + \frac{1}{\rho^2}[(r^2+M^2)^2 - (r-M)^2M^2 sin^2\theta]sin^2\theta d\phi_*^2 \\ &\quad -2M sin^2\theta dr d\phi_* - \frac{4M^2 r}{\rho}\sin^2\theta d\phi_* dv . \end{aligned} \tag{2.8}$$

Eliminating dr in favor of du and dv yields

$$\begin{aligned} ds^2 &= [\tfrac{(r-M)^2}{(r^2+M^2)} + (\tfrac{2M}{r} - 1)]dv^2 - \tfrac{(r-M)^2}{(r^2+M^2)}dudv \\ &\quad + \tfrac{1}{\rho^2}[(r^2+M^2)^2 - (r-M)^2M^2 sin^2\theta]sin^2\theta d\phi_*^2 \\ &\quad -[\tfrac{4M^2 r}{\rho^2} + \tfrac{M(r-M)^2}{(r^2+M^2)}]sin^2\theta d\phi_* dv + \tfrac{M(r-M)^2}{(r^2+M^2)}sin^2\theta d\phi_* du + \rho^2 d\theta^2 . \end{aligned} \tag{2.9}$$

This metric is evidently degenerate at $r = M$ and so the coordinates u, v are not good on the horizon at extremality. However, the $(r-M)^2$ degeneracy is of exactly the same form LRS found for the extremal RN case. Thus, the same coordinate extension should work here as well. We define a transformation

$$\psi(\xi) = 4M\left(\ln\xi - \frac{M}{2\xi}\right) \tag{2.10}$$

and assume

$$\begin{array}{lll} u = -\psi(-\mathcal{U}) & \Leftrightarrow & \mathcal{U} = -\psi^{-1}(u) \\ v = \psi(\mathcal{V}) & \Leftrightarrow & \mathcal{V} = \psi^{-1}(u) \end{array} \tag{2.11}$$

where $\mathcal{U}$ and $\mathcal{V}$ are the new coordinates that should be regular on the horizon at extremality.

To motivate this choice of ψ, recall that v, u are constructed adding or subtracting r_* to t. In the SD case, r_* has only a logarithmic divergence at $r = 2M$, as can be seen by setting $a = 0$ in (2.5). (The same holds for nonextremal RN). If one took only the first term in (2.10), $\psi(u) = 4M\ln(\mathcal{U})$, then one would have $\mathcal{U} = -e^{-\kappa u}$; $\kappa = 1/4M$, which is exactly the Kruskal transformation. However, for charged, rotating holes $\kappa \sim (M^2 - a^2 - Q^2)$. Thus, in the extremal limit, the surface gravity vanishes and, as mentioned above, the Kruskal transformation becomes ill-defined. From (2.7), however, we observe that at extremality, r_* has not only the logarithmic divergence of the nonextremal case but the added pole $(2M^2)/(r-M)$. We therefore pick the simplest possible generalization of the Kruskal transformation and add this term to ψ to get (2.10). Note also that $\psi'(\xi) = 4M/\xi + 2M^2/\xi^2 > 0$, always, and so ψ is monotonic; therefore (2.11) is a well-defined coordinate transformation. From (2.7) we have at extremality

$$r_*(M, M) = r + \frac{1}{2}\psi(r-M), \tag{2.12}$$

Near the horizon v is constant, which implies $t \sim -r_*$ and therefore $u \sim -2r* \sim -\psi(r-M)$. Thus, $\mathcal{U} = -\psi^{-1} \sim -(r-M)$ and the derivative $\psi'(-\mathcal{U}) \sim 4M^2/(r-M)^2$. It is this factor of $(r-M)^2$ in the denominator that will kill the $(r-M)^2$ factors in the numerator of (2.9), for when we rewrite the metric in terms of the new variables $\mathcal{U}, \mathcal{V}$ we find

$$\begin{aligned} ds^2 \approx\; & [\tfrac{(r-M)^2}{(r^2+M^2)} + (\tfrac{2M}{r} - 1)]\psi'(\mathcal{V})^2 d\mathcal{V}^2 - \tfrac{4M^2}{(r^2+M^2)}\psi'(\mathcal{V})d\mathcal{U}d\mathcal{V} \\ & + \tfrac{1}{\rho^2}[(r^2+M^2)2 - (r-M)^2M^2 sin^2\theta]sin^2\theta d\phi_*^2 \\ & -[\tfrac{4M^2 r}{\rho} + \tfrac{M(r-M)^2}{(r^2+M^2)}]sin^2\theta d\phi_*\psi'(\mathcal{V})d\mathcal{V} + \tfrac{4M^3}{(r^2+M^2)}d\phi_* d\mathcal{U} + \rho^2 d\theta^2 \end{aligned} \tag{2.13}$$

Note that since $\mathcal{V}$ is everywhere nonzero and finite, $\psi'(\mathcal{V})$ is regular on the horizon. Thus nothing particular happens at $r = M$ and ψ is a good coordinate transformation there. Note also that the degeneracy on the horizon is due solely to the behavior of the coordinate u. For this reason the coordinate extension ψ is good for any value of θ and the results are completely general.

3 Late-time history

We now use ψ to construct the late-time history of the collapsing star's center in the coordinates u, v. LRS performed such a calculation for the case of spherical symmetry. Of course the Kerr metric is only axially symmetric, not spherically symmetric, but if we specialize to the equatorial plane $\theta = \pi/2$, then the problem becomes two-dimensional and can be treated in the same way. Moreover, as long as we are concerned solely with the *center* of the star (as opposed to, say, the surface), all angular coordinates become degenerate and cannot be relevant. Thus, even for arbitrary θ, one evidently needs to consider only r and t (or u and v), as LRS did for the RN case.

The coordinates u and v are valid only outside the collapsing star and so the trajectory of the center, we must extend them to the interior. This is accomplished in standard fashion [6]. Ignoring the angular variables, the interior can be described by some interior null coordinates, U and V and the metric components $g_{\mu\nu}(U, V)$ can be chosen to be regular on the horizon. The center of the star can be taken at radial coordinate $= 0$, in which case $V = U$ and $\mathrm{d}V = \mathrm{d}U$ there.

Because the Kruskal-like coordinates $\mathcal{U}$ and $\mathcal{V}$ are regular everywhere as well, they can be matched to U and V. In particular, if two nearby outgoing rays differ by $\mathrm{d}U$ inside the star, then they will also differ by $\mathrm{d}U = \beta(\mathcal{U})\mathrm{d}\mathcal{U}$, with β a regular function, outside. By the same token, since V and v are regular everywhere, we have $\mathrm{d}V = \zeta(v)\mathrm{d}v$, where ζ is another regular function. In fact, if we consider the last ray $v = \bar{v}$ that passes through the center of the star before the formation of the horizon, then to first order $\mathrm{d}V = \zeta(\bar{v})\mathrm{d}v$, where $\zeta(\bar{v})$ is now constant.

We can write near the horizon

$$\mathrm{d}U = \beta(0)\frac{\mathrm{d}\mathcal{U}}{\mathrm{d}u}\mathrm{d}u \ . \tag{3.1}$$

Since for the center of the star $\mathrm{d}U = \mathrm{d}V = \zeta(\bar{v})\mathrm{d}v$, this immediately integrates to

$$\zeta(\bar{v})(v - \bar{v}) = \beta(0)\mathcal{U}(u) = -\beta(0)\psi^{-1}(-u) \sim -2\beta(0)\frac{M^2}{u} \ . \tag{3.2}$$

The last approximation follows from Eq. (2.10) where $\xi \sim \psi^{-1}(-2M^2/\xi)$ near the horizon.

Thus the late-time worldline for the center of the star is, finally, represented by the equation

$$v \sim \bar{v} - \frac{A}{u} \ , \quad u \to +\infty \ , \tag{3.3}$$

where $A = 2\beta(0)M^2/\zeta(\bar{v})$ is a positive constant that depends on the details of the internal metric and consequently on the dynamics of collapse.[2] This worldine is obviously hyperbolic and corresponds identically to that of a uniformly accelerated mirror in Minkowski space. This should be contrasted with the late-time worldline of a nonextremal hole, as can be found by using the "Kruskal-part" of the transformation ψ in (3.2).

$$v \sim \bar{v} - B\mathrm{e}^{-\kappa u} \ , \quad u \to +\infty \ . \tag{3.4}$$

[2]One might wonder why we did not consider $dV = \zeta(\mathcal{V})d\mathcal{V}$ instead of $\mathrm{d}V = \zeta(v)\mathrm{d}v$. In the former case we do not know the function $\zeta(\mathcal{V})$ and so the above integration could not be carried out.

As shown by LRS for the case of RN holes, the hyperbolic worldline cannot be obtained as an approximation of the nonextremal worldline (3.4) and so (3.3) represents a true discontinuity between extremal RN holes and their nonextremal counterparts. We now see the same conclusion applies to extremal Kerr holes.

With the benefit of hindsight this result is perhaps not so surprising, at least for the RN case. Consider the typical homework problem of dropping a rock into a black hole from some $r = R_o$. One generally solves this by starting from the condition on the four-velocity $u_\mu u^\mu = -1$ and the condition $u_o = -\tilde{E} =$ energy per unit mass $=$ constant. For RN one has $u_o = -(1 - 2M/r + Q^2/r^2)u^o$ and $u_r = (1 - 2m/R + Q^2/r^2)^{-1}u^r$, with $u^o = dt/d\tau$ and $u^r = dr/d\tau$. Forming dr/dt for the extremal RN metric, one gets after a few elementary substitutions

$$dt = \frac{Mz^3 dz}{(z-1)^3[z^2 - e^2]^{1/2}} \tag{3.5}$$

where $e \equiv \tilde{E}^{-1}$ and $z \equiv 1 + M/(r - M)$. The integral is cumbersome but can be evaluated analytically in terms of elementary functions. The result near the horizon ($z >> 1$) is

$$t = k_1 + Mz + 2M \ln z + \frac{k_2}{z} + \mathcal{O}(\frac{1}{z^2}) \tag{3.6}$$

where k_1 and k_2 are constants. For the extremal RN solution, $r_* = r - M^2/(r - M) + 2M \ln(r - M)$. Near the horizon ($r = M$) the M^2 term dominates. Using this and the fact that in the same regime $t \approx -r_* \Rightarrow u \approx -2r_*$, one brings the above equation into the form

$$v \approx k_1 + \frac{k_3}{u}, \tag{3.7}$$

again a hyperbolic worldline, as distinct from the exponential form similar to (3.4) that one finds for a rock falling into a SD or nonextremal RN black hole. As mentioned in the Introduction, the horizon structure changes completely at extremality. Moreover, the horizon itself is moved in to $r = M$, and the shape of the effective potential is totally altered. Given all this, it should really come as no suprise that the trajectories of objects falling into extremal versus nonextremal holes should differ.

4 Nonthermal Spectrum

To implement the moving mirror analogy, one requires that the wave equation $\phi_{;\mu}{}^{;\mu} = 0$ be separable, which implies that $\phi = f(u) + g(v)$. Then one move the problem to Minkowski space, where the null coordinates are now assumed to be $u = t - x$ and $v = t + x$. The mirror trajectory is taken to be that of the center of the star, $v = p(u)$. With the boundary conditions that one has pure ingoing solutions on $\mathcal{I}^-$ and pure outgoing solutions on $\mathcal{I}^+$ and that there be total reflection at the mirror, i.e., $\phi(p(u), u) = 0$, it is easy to show that

$$\phi_\omega^{(\mathrm{in})}(u, v) = \frac{\mathrm{i}}{\sqrt{4\pi\omega}} \left(\mathrm{e}^{-\mathrm{i}\omega v} - \mathrm{e}^{-\mathrm{i}\omega p(u)}\right) \tag{4.1}$$

and

$$\phi_\omega^{(\mathrm{out})}(u, v) = \frac{\mathrm{i}}{\sqrt{4\pi\omega}} \left(\mathrm{e}^{-\mathrm{i}\omega u} - \Theta\left(\bar{v} - v\right)\mathrm{e}^{-\mathrm{i}\omega q(v)}\right) \ . \tag{4.2}$$

Here $q(v) = p^{-1}(v)$ and $\omega > 0$. (See LRS for further details.)

The relevant Bogoliubov coefficient is defined as

$$\beta_{\omega\omega'} = -\left(\phi^{(\mathrm{out})}_{\omega}, \phi^{(\mathrm{in})*}_{\omega'}\right) = \mathrm{i}\int_0^{+\infty} \mathrm{d}x \left[\phi^{(\mathrm{out})}_{\omega}(u,v)\overleftrightarrow{\partial_t}\,\phi^{(\mathrm{in})}_{\omega'}(u,v)\right]_{t=0} . \tag{4.3}$$

which gives a particle spectrum

$$\langle N_\omega \rangle = \int_0^{+\infty} \mathrm{d}\omega' \, |\beta_{\omega\omega'}|^2 . \tag{4.4}$$

For SD and nonextremal RN, it is the evaluation of (4.3) and (4.4) that yields the black-body spectrum because the function p is exponential at late times. For the extremal RN case, however, one uses the hyperbolic form, equivalent to a uniformly accelerated mirror. The properties of the uniformly accelerated mirror have been extensively studied and the Bogoliubov coefficients have been calculated [6]. In the asymptotic regime for the trajectory (3.3)

$$\beta_{\omega\omega'} \approx \frac{\sqrt{A}}{\pi} \mathrm{e}^{\mathrm{i}\sqrt{A}(\omega-\omega')} K_1(2(A\omega\omega')^{1/2}) , \tag{4.5}$$

where K_1 is a modified Bessel function. For argument z, $K_1(z) \sim 1/z$ for $z \to 0$, and $K_1(z) \sim \sqrt{\pi/(2z)}\,\mathrm{e}^{-z}$ when $z \to +\infty$ [7]

For the extremal Kerr black hole the analysis goes through almost unchanged. The solutions to wave equation for Kerr can in fact be separated into the above form [8], where now u and v are exactly the null coordinates we have been using. In the vicinity of the Kerr horizon, the radial wave function has the form $R \sim exp(\pm i\tilde{\omega} r_*)$. Here, $\tilde{\omega} \equiv \omega - m\Omega$, m is the azimuthal quantum number and $\Omega = 1/(2M)$ is the angular velocity of the black hole on the horizon at extremality. The outgoing modes $\phi^{(out)} \sim exp(-i\tilde{\omega}u)$ get transported back to to become $\phi^{(out)} \sim exp(i\tilde{\omega}q(v))$ on $\mathcal{I}^-$. The Bogoliubov coefficient is essentially the Fourier transform of this quantity,

$$\beta_{\tilde{\omega}'\tilde{\omega}} \approx \int exp\{i\tilde{\omega}'v + i\tilde{\omega}q(v)\}dv \tag{4.6}$$

which is the relevant integral one encounters while evaluating (4.3). Thus, since we have established that $p(u)$ and hence $q(v)$ are the same for both extremal Kerr and extremal RN, the Bogoliubov coefficients are also the same with the replacement $\omega \to \tilde{\omega}$. This gives the frequency shift mentioned in the Introduction. Consequently, with the substitution $\tilde{\omega}$ for ω, the extremal Kerr spectrum is found to be identical to the extremal RN spectrum.

The main point is that since the spectrum goes like the square of a K_1 Bessel function it is definitely not a black-body spectrum. Therefore temperature is simply undefined. Moreover, as shown by LRS the variance of the stress-energy tensor decays to zero as the power law

$$\langle : T^2_{uu} : \rangle = -\frac{A^2}{4\pi^2 \left(A + (v-\bar{v})\,u\right)^4} \sim -\frac{A^2}{4\pi^2 \left(v-\bar{v}\right)^4 u^4} . \tag{4.7}$$

as distinct from the nonextremal black hole, the variance of whose stress-energy tensor decays exponentially:

$$\langle : T^2_{uu} : \rangle = \frac{1}{(4\pi)^2}\left(\frac{\kappa^4}{48} - \frac{4\kappa^2 B^2 \mathrm{e}^{-2\kappa u}}{\left(v - \bar{v} + B\mathrm{e}^{-\kappa u}\right)^4}\right) \sim \frac{\kappa^4}{768\pi^2} - \frac{\kappa^2 B^2 \mathrm{e}^{-2\kappa u}}{4\pi^2 \left(v-\bar{v}\right)^4} . \tag{4.8}$$

Contrary to the extremal case, the “nonextremal” variance ΔT_{uu} tends not to zero as $u \to +\infty$, but to the value $\kappa^2\sqrt{2}/(48\pi)$, which corresponds to thermal emission.

5 Conclusions

All these results, including those of [2], demonstrate that at no time in their history do extremal black holes act as thermal objects . The conclusion would seem to have significant implications in regard to string theory. From the point of view of thermodynamics, if temperature is undefined, entropy is also undefined. Yet string caclulations [9], which involve only state-counting arguments, indicate that extremal black hole solutions possess the usual Bekenstein-Hawking entropy. The results of semi-classical gravity and string theory would appear, then, to be in direct contradiction. The resolution to this dilemma is far from clear. The calculations discussed in this letter assume $a = M$ configurations exist, but ignore back-reaction to the metric (or recoil of the mirror). Evidently, extraordinary fine-tuning early on is required to produce extremal solutions and it may well be that, taking such backreaction into account, the formation of extremal objects is simply disallowed by nature. If true, string-theory calculations refer to impossible objects. On the other hand, if one cannot ignore back-reaction—at least at late times—then all semi-classical calculations to date (including Hawking's) are incorrect. Presently, about all that can be said is that extremal black holes seem to represent the limits of our current theories of quantum gravity.

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